SECONDARY SCHOOL IMPROVEMENT PROGRAMME (SSIP) 2019

GAUTENG PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

GRADE 12

SUBJECT: MATHEMATICS

LEARNER NOTES

(PAGE 1 OF 16)

SESSION NO: 4

TOPIC: EUCLIDEAN GEOMETRY (REVISION OF GR 11 CIRCLE GEOMETRY)

SECTION A: TYPICAL EXAM QUESTIONS

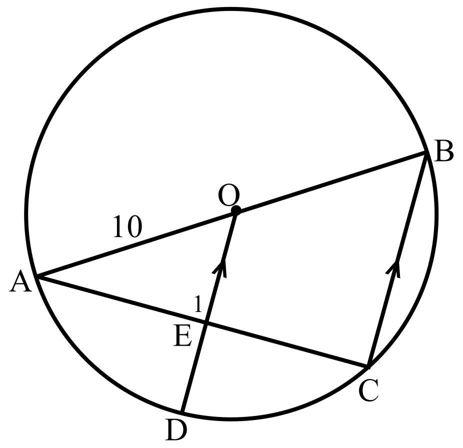
QUESTION 1

1.1 Complete:

1.1.1 The angle in a semi-circle is ……. (1)

1.1.2 The perpendicular from the centre of a circle to a chord …….. (1)

1.2 In the diagram below, AOB is a diameter of the circle centre O. OD||BC. The length of the radius is 10 units.



1.2.1 What is the size of $\hat{C}$? State a reason. (2)

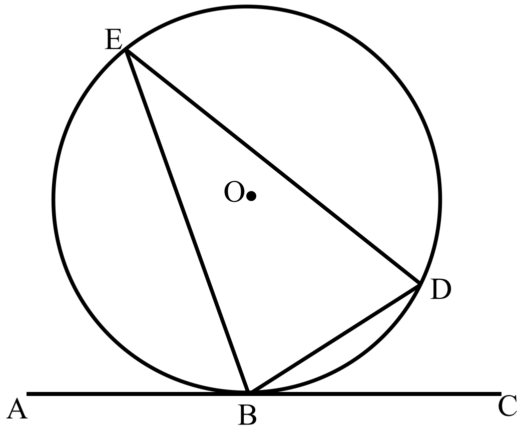
1.2.2 What is the size of $\hat{E}_1$? State a reason. (2)

1.2.3 Why is $AE = EC$? State a reason. (1)

1.2.4 If $AC = 16$ units, calculate the length of ED. (3)

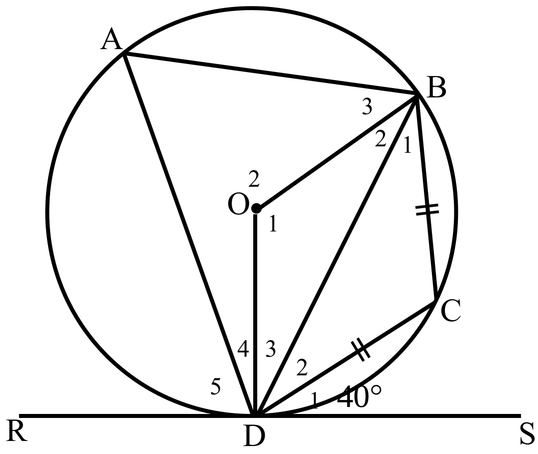
QUESTION 2

2.1 Prove the theorem that states that the angle between a tangent and a chord is equal to the angle in the alternate segment. Use the diagram below to do the necessary construction. (6)



2.2 In the figure below, RDS is a tangent to the circle centre O at D.

BC = DC and $C\hat{D}S = 40°$

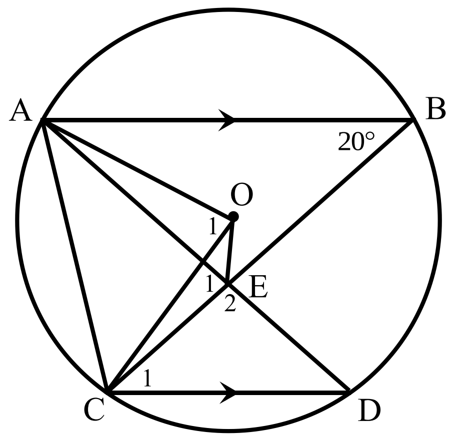


2.2.1	What is the size of $\hat{B}_1$. State a reason.	(2)
2.2.2	What is the size of $\hat{D}_2$. State a reason.	(2)
2.2.3	What is the size of $\hat{C}$. State a reason.	(2)
2.2.4	Calculate the size of $\hat{O}_2$ State a reason.	(2)
2.2.5	Calculate the size of $\hat{O}_1$. State a reason.	(2)
2.2.6	Calculate the size of $\hat{D}_3$ State reasons.	(3)
2.2.7	Calculate the size of $\hat{A}$ State a reason.	(2)

QUESTION 3

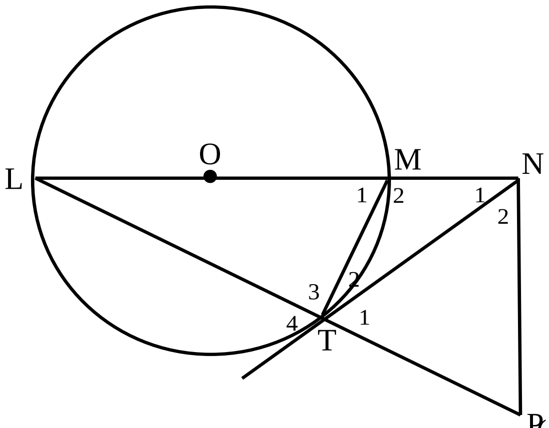
In the diagram, O is the centre of the circle passing through A, B, C and D.

AB||CD and $\hat{B} = 20°$



3.1	Calculate the size of $\hat{C}_1$? State a reason.	(2)
3.2	Calculate the size of $\hat{O}_1$? State a reason.	(2)
3.3	Calculate the size of $\hat{D}$? State a reason.	(2)
3.4	Calculate the size of $\hat{E}_1$? State a reason.	(2)
3.5	Why is AOEC a cyclic quadrilateral?	(1)

QUESTION 4



LOM is a diameter of circle LMT. The centre is O. TN is a tangent at T. $LN \perp NP$.
MT is a chord. LT is a chord produced to P.

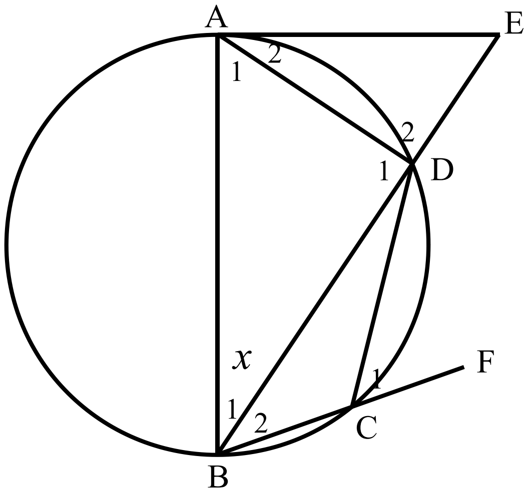
Prove that:

4.1 MNPT is a cyclic quadrilateral (4)

4.2 NP = NT (5)

QUESTION 5

In the diagram below, AB is a diameter of the circle ABCD. AE is a tangent to the circle at A. $\hat{B}_1 = x$.



5.1 Prove that AB is a tangent to the circle through A, D and E. (7)

5.2 Prove that $\hat{C}_1 = \hat{E}$ (2)

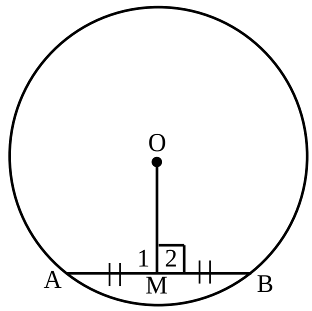
SECTION B: NOTES ON CONTENT

REVISION OF GRADE 11 THEOREMS

THEOREM

The line segment joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

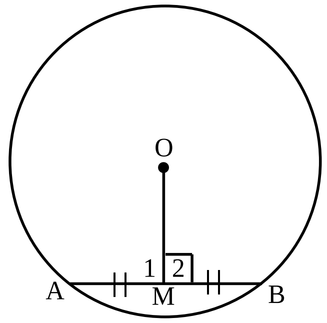
If $AM = MB$ then $OM \perp AB$ which means that $\hat{M}_1 = \hat{M}_2 = 90°$



THEOREM CONVERSE

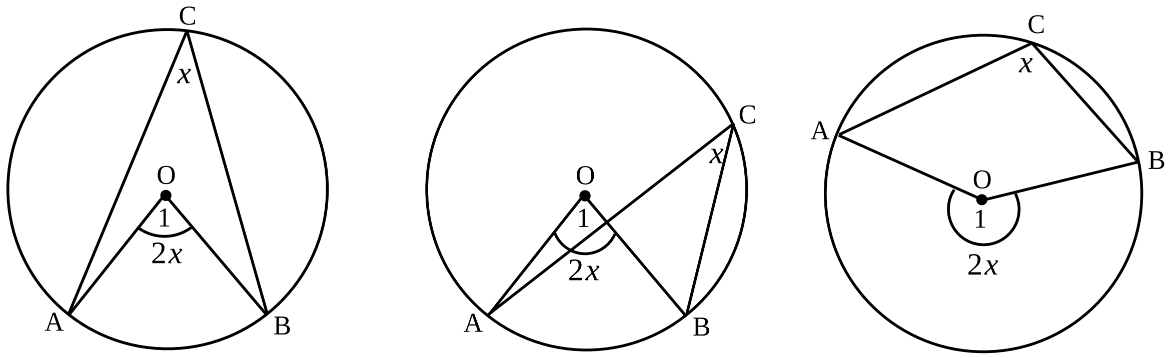
The perpendicular drawn from the centre of a circle to a chord bisects the chord.

If $OM \perp AB$ then $AM = MB$



THEOREM

The angle which an arc of a circle subtends at the centre of a circle is twice the angle it subtends at the circumference of the circle.

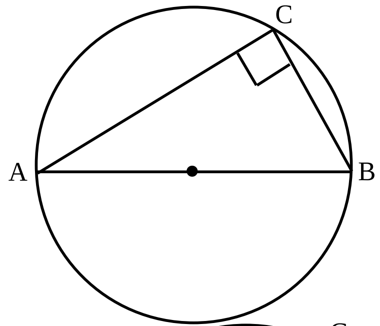


For all three diagrams: $\hat{O}_1 = 2\hat{C}$

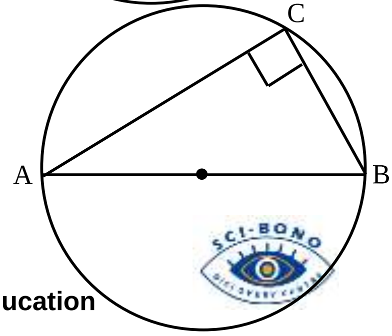
THEOREM

The angle subtended at the circle by a diameter is a right angle. We say that the angle in a semi-circle is $90°$.

In the diagram $\hat{C} = 90°$



THEOREM CONVERSE



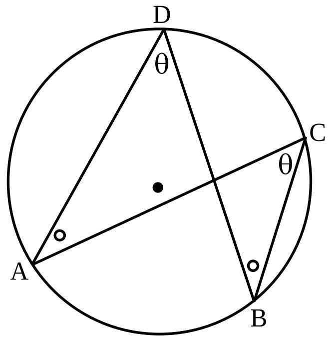
If the angle subtended by a chord at a point on the circle is $90°$, then the chord is a diameter.

If $\hat{C} = 90°$, then the chord subtending $\hat{C}$ is a diameter.

THEOREM

An arc or line segment of a circle subtends equal angles at the circumference of the circle. We say that the angles in the same segment of the circle are equal.

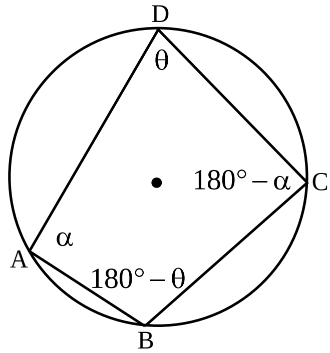
In the diagram, $\hat{A} = \hat{B}$ and $\hat{C} = \hat{D}$



THEOREM

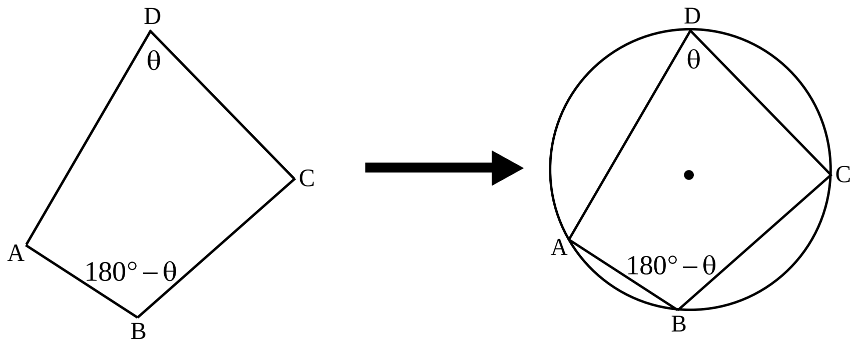
The opposite angles of a cyclic quadrilateral are supplementary (add up to $180°$)

In the diagram, $\hat{A} + \hat{C} = 180°$ and $\hat{B} + \hat{D} = 180°$



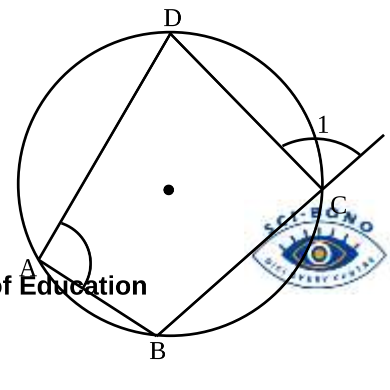
THEOREM CONVERSE

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is a cyclic quadrilateral.



THEOREM

An exterior angle of a cyclic quadrilateral is equal to

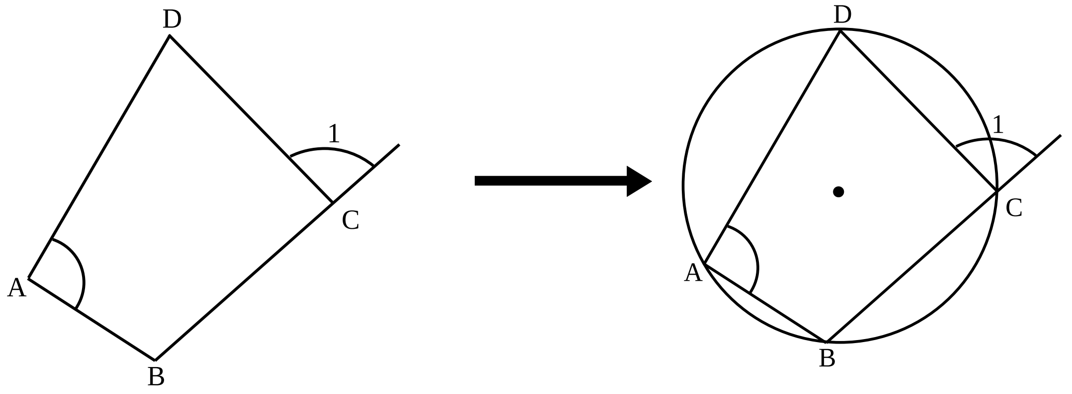


the interior opposite angle.

If ABCD is a cyclic quadrilateral, then $\hat{C}_1 = A$.

<u>THEOREM CONVERSE</u>

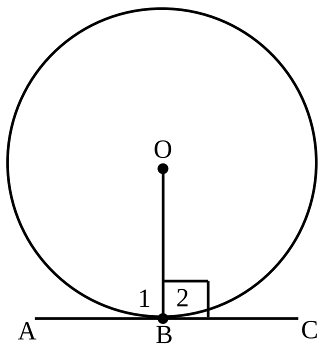
If an exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is a cyclic quadrilateral.



<u>THEOREM</u>

A tangent to a circle is perpendicular to the radius at the point of contact.

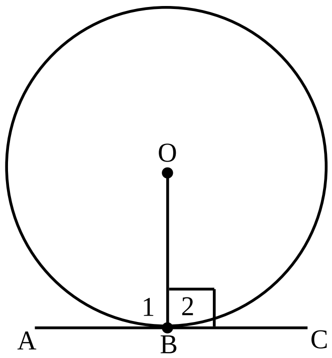
If ABC is a tangent to the circle at the point B, then the radius $OB \perp ABC$, i.e. $\hat{B}_1 = \hat{B}_2 = 90°$.



<u>THEOREM CONVERSE</u>

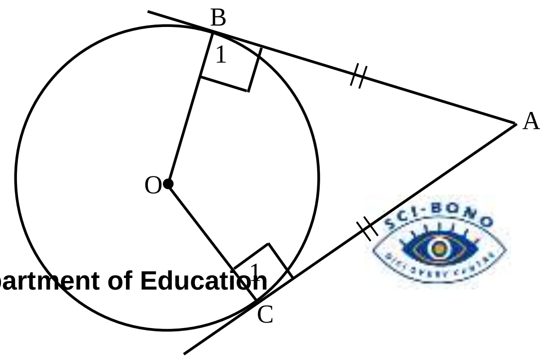
If a line is drawn perpendicular to a radius at the point where the radius meets the circle, then the line is a tangent to the circle.

If line $ABC \perp OB$ and if OB is a radius, then ABC is a tangent to the circle at B.



<u>THEOREM</u>

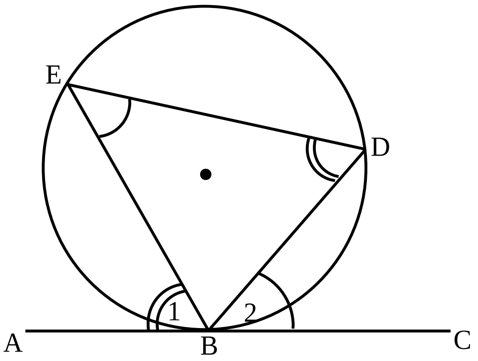
If two tangents are drawn

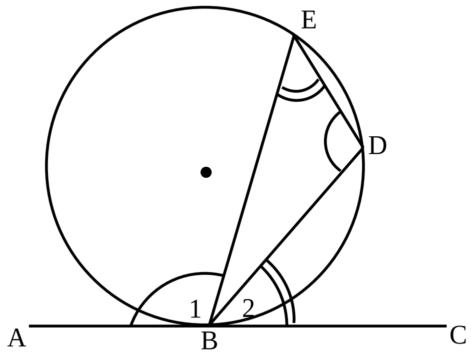


from the same point outside a circle, then they are equal in length.

THEOREM

The angle between a tangent to a circle and a chord drawn from the point of contact in equal to an angle in the alternate segment.

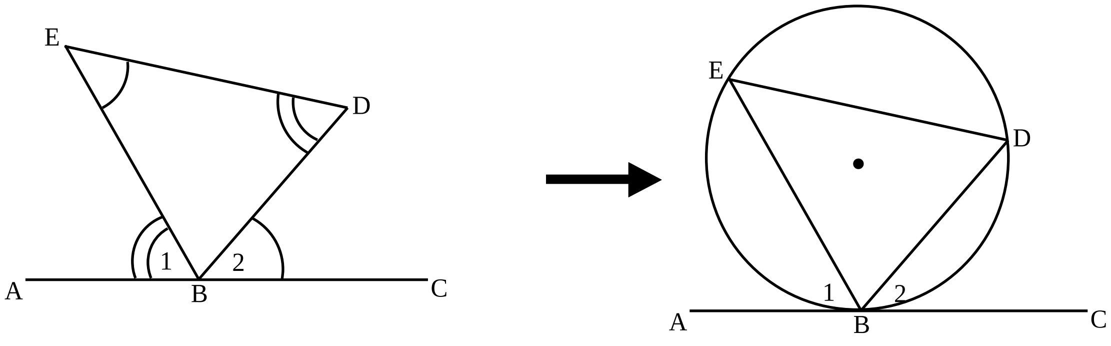




In both diagrams, $\hat{B}_2 = \hat{E}$ and $\hat{B}_1 = \hat{D}$.

THEOREM CONVERSE

If a line is drawn through the endpoint of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.



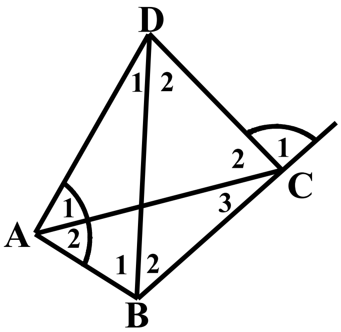
If $\hat{B}_2 = \hat{E}$ or if $\hat{B}_1 = \hat{D}$, then ABC is a tangent to the circle passing through the points B, D and E.

How to prove that a quadrilateral is cyclic

ABCD will be a cyclic quadrilateral if one of the following conditions is satisfied.

Condition 1

$(\hat{A}_1 + \hat{A}_2) + (\hat{C}_2 + \hat{C}_3) = 180°$ or $(\hat{B}_1 + \hat{B}_2) + (\hat{D}_1 + \hat{D}_2) = 180°$



Condition 2

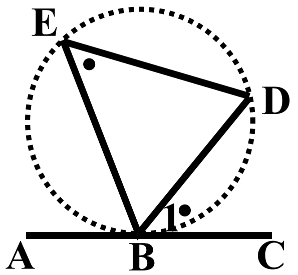
$\hat{C}_1 = \hat{A}_1 + \hat{A}_2$

Condition 3

$\hat{D}_1 = \hat{C}_3$ **or** $\hat{D}_2 = \hat{A}_2$ **or** $\hat{C}_2 = \hat{B}_1$ **or** $\hat{B}_2 = \hat{A}_1$

<u>How to prove that a line is a tangent to a circle</u>

ABC would be a tangent to the "imaginary" circle drawn through EBD if $\hat{B}_1 = \hat{E}$



<u>PROOFS OF THEOREMS REQUIRED FOR EXAMS</u>

<u>THEOREM</u>

The line drawn from the centre of a circle perpendicular to a chord bisects the chord.

<u>Proof</u>

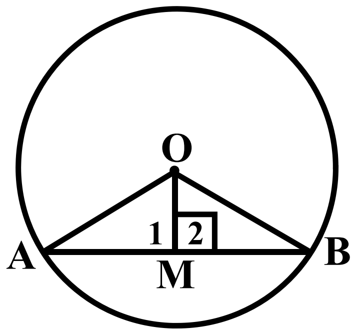
Join OA and OB.

In ΔOAM and ΔOBM:

(a)	OA = OB	radii
(b)	$\hat{M}_1 = \hat{M}_2 = 90°$	given
(c)	OM = OM	common

$\therefore \Delta OAM \equiv \Delta OBM$ RHS

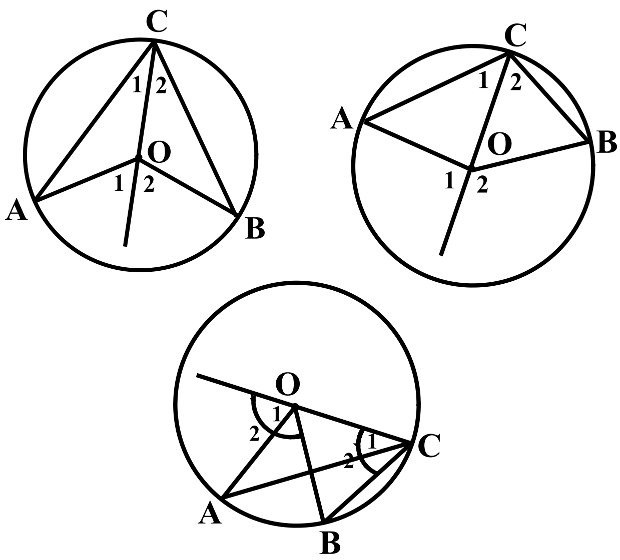
$\therefore$ AM = MB



<u>THEOREM</u>

The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre).

<u>Proof</u>



Join CO and produce.
For all three diagrams:

$\hat{O}_1 = \hat{C}_1 + \hat{A}$ ext $\angle$ of ΔOAC

But $\hat{C}_1 = \hat{A}$ OA = OC , radii

$\therefore \hat{O}_1 = 2\hat{C}_1$

Similarly, in ΔOCB $\hat{O}_2 = 2\hat{C}_2$

$\therefore \hat{O}_1 + \hat{O}_2 = 2\hat{C}_1 + 2\hat{C}_2$

$\therefore \hat{O}_1 + \hat{O}_2 = 2(\hat{C}_1 + \hat{C}_2)$

$\therefore A\hat{O}B = 2A\hat{C}B$

For the third diagram:

$\therefore \hat{O}_2 - \hat{O}_1 = 2\hat{C}_2 - 2\hat{C}_1$

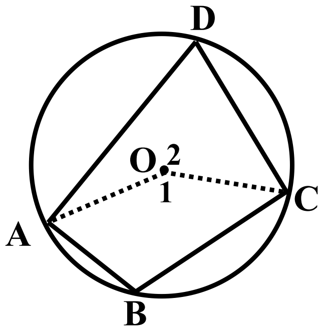
$\therefore \hat{O}_2 - \hat{O}_1 = 2(\hat{C}_2 - \hat{C}_1)$

$\therefore A\hat{O}B = 2A\hat{C}B$

<u>THEOREM</u>

The opposite angles of a cyclic quadrilateral are supplementary (add up to $180°$).

<u>Proof</u>



Join AO and OC.

$\hat{O}_1 = 2\hat{D}$ $\angle$ at centre = $2 \times \angle$ at circ

$\hat{O}_2 = 2\hat{B}$ $\angle$ at centre = $2 \times \angle$ at circ

$\hat{O}_1 + \hat{O}_2 = 2\hat{D} + 2\hat{B}$

and $\hat{O}_1 + \hat{O}_2 = 360°$ $\angle$'s round a point

$\therefore 360° = 2(\hat{D} + \hat{B})$

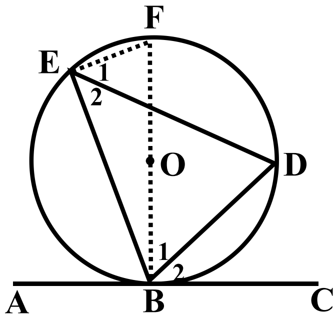
$\therefore 180° = \hat{D} + \hat{B}$

Similarly, by joining BO and DO, it can be proven that $\hat{A} + \hat{C} = 180°$

<u>THEOREM</u>

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to the angle in the alternate segment.

Proof



Draw diameter BOF and join EF

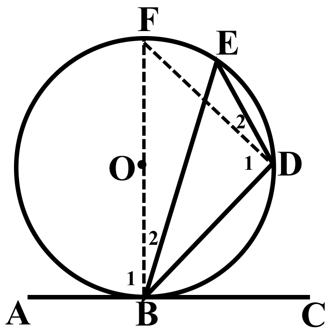
$\hat{B}_1 + \hat{B}_2 = 90°$ tan ⊥ rad

$\hat{E}_1 + \hat{E}_2 = 90°$ ∠ in a semi-circle

But $\hat{B}_1 = \hat{E}_1$ arc FD subt = ∠'s

$\therefore \hat{B}_2 = \hat{E}_2$

$\therefore C\hat{B}D = B\hat{E}D$



Draw diameter BOF and join FD

$\hat{B}_1 = 90°$ tan ⊥ rad

$\hat{D}_1 = 90°$ ∠ in a semi-circle

$\hat{B}_2 = \hat{D}_2$ arc FE subt = ∠'s

Now $\hat{B}_1 = \hat{D}_1 = 90°$ and $\hat{B}_2 = \hat{D}_2$

$\therefore \hat{B}_1 + \hat{B}_2 = \hat{D}_1 + \hat{D}_2$

$A\hat{B}E = B\hat{D}E$

SECTION C: HOMEWORK QUESTIONS

QUESTION 1

In the diagram below, M is the centre of the circle. D, E, F and G are points on the circle. If $\hat{F}_1 = 10°$ and $\hat{D}_2 = 50°$, calculate, with reasons, the size of:

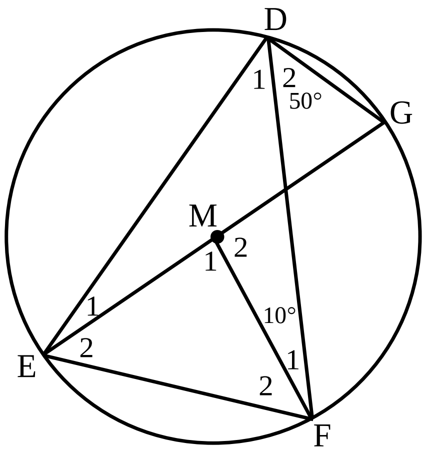
1.1 $\hat{D}_1$ (2)

1.2 $\hat{M}_1$ (2)

1.3 $\hat{F}_2$ (2)

1.4 $\hat{G}$ (2)

1.5 $\hat{E}_1$ (2)



QUESTION 2

In the diagram below, QP is a tangent to a circle with centre O. RS is a diameter of the circle and RQ is a straight line. T is a point on the circle. PS bisects $T\hat{P}Q$ and $S\hat{P}Q = 22°$. Calculate the following, giving reasons:

2.1 $\hat{P}_2$ (2)

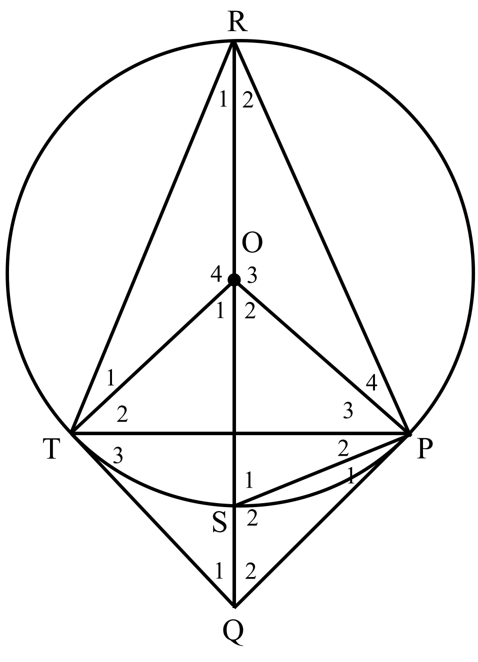
2.2 $\hat{R}_2$ (2)

2.3 $\hat{P}_3 + \hat{P}_4$ (3)

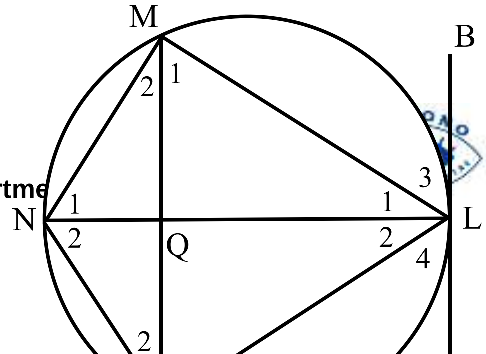
2.4 $\hat{R}_1$ (4)

2.5 $\hat{O}_1$ (3)

2.6 $\hat{Q}_2$ (3)



QUESTION 3



ALB is a tangent to circle LMNP. ALB||MP.
Prove that:

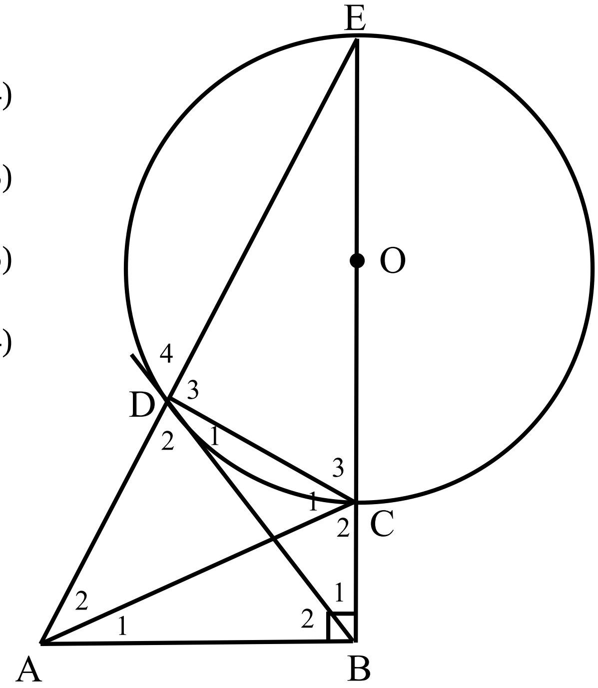
3.1 LM = LP (4)

3.2 LN bisects $M\hat{N}P$ (4)

3.3 LM is a tangent to circle MNQ (4)

QUESTION 4

EC is a diameter of circle DEC. EC is produced to B. BD is a tangent at D. ED is produced to A and $AB \perp BE$.
Prove that:



4.1 ABCD is a cyclic quadrilateral. (4)

4.2 $\hat{A}_1 = \hat{E}$ (3)

4.3 $BD = BA$ (5)

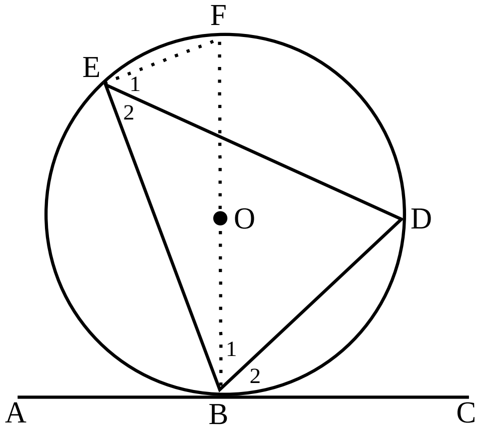
4.4 $\hat{C}_2 = \hat{C}_3$ (4)

SECTION D: SOLUTIONS FOR SECTION A

QUESTION 1

1.1.1	equal to $90°$	✓ answer (1)
1.1.2	bisects the chord	✓ answer (1)
1.2.1	$\hat{C} = 90°$ Angle in semi-circle	✓ answer ✓ reason (2)
1.2.2	$\hat{E}_1 = 90°$ Corr angles equal	✓ answer ✓ reason (2)
1.2.3	Perp from centre to chord	✓ answer (1)
1.2.4	$AE = 8$ units Given $\therefore OE^2 = 10^2 - 8^2$ $\therefore OE = 6$ units But $OD = AO = 10$ units $\therefore ED = 4$ units	✓ $AE = 8$ units ✓ $OE = 6$ units ✓ $ED = 4$ units (3)

QUESTION 2

2.1	 Draw diameter BOF and join EF $\hat{B}_1 + \hat{B}_2 = 90°$ ….. tan ⊥ rad $\hat{E}_1 + \hat{E}_2 = 90°$ ….. ∠ in semi-circle But $\hat{B}_1 = \hat{E}_1$ ….. FD subt = ∠s $\therefore \hat{B}_2 = \hat{E}_2$	✓ construction ✓ $\hat{B}_1 + \hat{B}_2 = 90°$ ✓ $\hat{E}_1 + \hat{E}_2 = 90°$ ✓ $\hat{B}_1 = \hat{E}_1$ ✓ $\hat{B}_2 = \hat{E}_2$ ✓ reasons (6)
2.2.1	$\hat{B}_1 = 40°$ Tan-chord	✓ $\hat{B}_1 = 40°$ ✓ reason (2)

2.2.2	$\hat{D}_2 = 40°$ Angles opp equal sides	✓ $\hat{D}_2 = 40°$ ✓ reason (2)
2.2.3	$\hat{C} = 100°$ Sum of the $\angle$'s of a Δ	✓ $\hat{C} = 100°$ ✓ reason (2)
2.2.4	$\hat{O}_2 = 200°$ $\angle$ at centre $= 2\times\angle$ at circle	✓ $\hat{O}_2 = 200°$ ✓ reason (2)
2.2.5	$\hat{O}_1 = 160°$ $\angle$'s round a point	✓ $\hat{O}_1 = 160°$ ✓ reason (2)
2.2.6	$\hat{D}_3 + \hat{B}_2 + \hat{O}_1 = 180°$ Sum of the $\angle$'s of a Δ $\therefore \hat{D}_3 + \hat{B}_2 + 160° = 180°$ $\therefore \hat{D}_3 + \hat{B}_2 = 20°$ But $\hat{D}_3 = \hat{B}_2$ Angles opp equal radii $\therefore \hat{D}_3 + \hat{D}_3 = 20°$ $\therefore 2\hat{D}_3 = 20°$ $\therefore \hat{D}_3 = 10°$	✓ $\hat{D}_3 + \hat{B}_2 + \hat{O}_1 = 180°$ ✓ $\hat{D}_3 = \hat{B}_2$ ✓ $\hat{D}_3 = 10°$ (3)
2.2.7	$\hat{A} = 80°$ Opp $\angle$'s cyclic quad or $\angle$ at centre $= 2\times\angle$ at circle	✓ $\hat{A} = 80°$ ✓ reason (2)

QUESTION 3

3.1	$\hat{C}_1 = 20°$ Alt angles equal	✓ $\hat{C}_1 = 20°$ ✓ reason (2)
3.2	$\hat{O}_1 = 40°$ $\angle$ at centre $= 2\times\angle$ at circle	✓ $\hat{O}_1 = 40°$ ✓ reason (2)
3.3	$\hat{D} = 20°$ $\angle$ at centre $= 2\times\angle$ at circle or Angles in same segment	✓ $\hat{D} = 20°$ ✓ reason (2)
3.4	$\hat{E}_1 = 40°$ Ext$\angle$ of triangle	✓ $\hat{E}_1 = 40°$ ✓ reason (2)
3.5	$\hat{E}_1 = \hat{O}_1 = 40°$	✓ answer (1)

QUESTION 4

4.1	$\hat{N}_1 + \hat{N}_2 = 90°$ Given $\hat{T}_3 = 90°$ $\angle$ in semi-circle $\therefore \hat{N}_1 + \hat{N}_2 = \hat{T}_3$ $\therefore$ MNPT is a cyclic quad Ext $\angle$ = int opp $\angle$	✓ $\hat{N}_1 + \hat{N}_2 = 90°$ ✓ $\hat{T}_3 = 90°$ ✓ $\hat{N}_1 + \hat{N}_2 = \hat{T}_3$ ✓ reasons (4)
4.2	$\hat{T}_1 = \hat{T}_4$ Vertically opp angles $\hat{T}_4 = \hat{M}_1$ Tan chord $\hat{M}_1 = \hat{P}$ Ext $\angle$ of cyclic quad $\therefore \hat{T}_1 = \hat{P}$ $\therefore$ NP = NT Sides opp equal $\angle$ s	✓ $\hat{T}_1 = \hat{T}_4$ ✓ $\hat{T}_4 = \hat{M}_1$ ✓ $\hat{M}_1 = \hat{P}$ ✓ $\hat{T}_1 = \hat{P}$ ✓ reasons (5)

QUESTION 5

5.1	$\hat{A}_1 + \hat{A}_2 = 90°$ Tan $\perp$ radius $\hat{A}_2 = x$ Tan-chord $\therefore \hat{A}_1 + x = 90°$ $\therefore \hat{A}_1 = 90° - x$ $\hat{A}_1 + \hat{A}_2 + \hat{B}_1 + \hat{E} = 180°$ Sum of the $\angle$'s of a Δ $\therefore 90° + x + \hat{E} = 180°$ $\therefore \hat{E} = 90° - x$ $\therefore \hat{A}_1 = \hat{E}$ $\therefore$ AB is a tangent to circle ADE since $\angle$ between line and chord equals $\angle$ in alt segment.	✓ $\hat{A}_1 + \hat{A}_2 = 90°$ ✓ $\hat{A}_2 = x$ ✓ $\hat{A}_1 = 90° - x$ ✓ $\hat{A}_1 + \hat{A}_2 + \hat{B}_1 + \hat{E} = 180°$ ✓ $\hat{E} = 90° - x$ ✓ $\hat{A}_1 = \hat{E}$ ✓ reasons (7)
5.2	$\hat{C}_1 = \hat{A}_1$ Ext $\angle$ of cyclic quad $\hat{A}_1 = \hat{E} = 90° - x$ Proved $\therefore \hat{C}_1 = \hat{E}$	✓ $\hat{C}_1 = \hat{A}_1$ ✓ $\hat{A}_1 = \hat{E} = 90° - x$ (2)